

# Pointwise estimates for solutions of fractal Burgers equation<sup>☆</sup>

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## Abstract

In this paper, we provide two-sided estimates and uniform asymptotics for the solution of  $d$ -dimensional critical fractal Burgers equation  $u_t - \Delta^{\alpha/2} u + b \cdot \nabla (u|u|^q) = 0$ ,  $\alpha \in (1, 2)$ ,  $b \in \mathbb{R}^d$  for  $q = (\alpha - 1)/d$  and  $u_0 \in L^1(\mathbb{R}^d)$ . We consider also  $q > (\alpha - 1)/d$  under additional condition  $u_0 \in L^\infty(\mathbb{R}^d)$ . In both cases we assume  $u_0 \geq 0$ , which implies that the solution is non-negative. The estimates are given in the terms of the function  $P_t u_0$ , where  $P_t$  is the stable semigroup operator.

*Keywords:* generalized Burgers equation, fractional Laplacian, estimates of solutions, asymptotics of solutions

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## 1. Introduction

Let  $d \in \mathbb{N}$ ,  $\alpha \in (1, 2)$  and  $q_0 = (\alpha - 1)/d$ . The goal of the paper is to describe estimates and asymptotics of solutions of the fractal Burgers equation

$$\begin{cases} u_t - \Delta^{\alpha/2} u + b \cdot \nabla (u|u|^q) = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where  $q \geq q_0$  and  $b \in \mathbb{R}^d$  is a constant vector. We assume that  $u_0 \in L^1$  and  $u_0 \geq 0$ , cf. (1.3), (1.4). Then, the solution  $u(t, x)$  is also non-negative and the absolute value in (1.1) may be omitted. Furthermore, the pseudo-differential operator  $\Delta^{\alpha/2}$  is the fractional Laplacian defined by the Fourier transform

$$\widehat{\Delta^{\alpha/2} \phi}(\xi) = -|\xi|^\alpha \widehat{\phi}(\xi), \quad \phi \in C_c^\infty(\mathbb{R}^d).$$

We denote the heat kernel related to this operator by  $p(t, x)$ . It is the fundamental solution of

$$v_t = \Delta^{\alpha/2} v. \quad (1.2)$$

The corresponding semigroup operator  $P_t$  is given by

$$P_t f(x) = \int_{\mathbb{R}^d} p(t, x - y) f(y) dy.$$

Linear and nonlinear gradient perturbations of fractional Laplacian have been intensely studied in recent years, e.g. [15, 17, 22, 6, 20, 10, 21, 11, 8]. Equation (1.1) was recently investigated in [2, 4, 3, 7] for various values of  $q$  and initial conditions  $u_0$ . For  $d = 1$ , the case  $q = 2$  is of particular interest (see e.g. [18, 1, 19, 23]) because it is a natural counterpart of the classical Burgers equation. In [4] the authors studied the solution of (1.1) for  $q = q_0$  and  $u_0 = M\delta_0$ , where  $\delta_0$  is the Dirac measure at 0 and  $M > 0$  is some constant. They showed the existence of the solution  $U_M(t, x)$  and its basic properties. In [7] pointwise estimates of  $U_M(t, x)$  were derived for small values of  $M$ . More precisely, it was proved that for sufficiently small  $M$ ,

$$0 \leq U_M(t, x) \leq c p(t, x), \quad t > 0, x \in \mathbb{R}^d,$$

for some constant  $c > 0$ . This result was improved in the recent paper [16]. The authors showed that for every  $M > 0$ , there is a constant  $c > 0$  such that the following estimates hold

$$c^{-1}p(t, x) \leq U_M(t, x) \leq cp(t, x), \quad t > 0, x \in \mathbb{R}^d.$$

The aim of this paper is to obtain similar results for  $u_0$  satisfying either of the following conditions, which depend on the value of  $q$ :

$$\bullet u_0 \in L^1(\mathbb{R}^d), \quad u_0 \geq 0, \quad \text{for } q = q_0, \quad (1.3)$$

$$\bullet u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad u_0 \geq 0, \quad \text{for } q > q_0. \quad (1.4)$$

Additionally, we assume throughout the paper that  $\|u_0\|_1 = M > 0$ . The value  $q_0$  is called a critical exponent. In this case linear and non-linear operators are balanced, whereas the fractional Laplacian is dominating for  $q > q_0$ . More precisely, the large time behaviour of the solution for  $q > q_0$  coincides with behaviour of the solution  $(P_t u_0)(x)$  of (1.2) ([3], Theorem 4.1)

$$\lim_{t \rightarrow \infty} t^{n(1-1/p)/\alpha} \|u(t, \cdot) - (P_t u_0)(\cdot)\|_p = 0, \quad \text{for each } p \in [1, \infty]. \quad (1.5)$$

On the other hand, for  $q = q_0$  the large time behaviour of the solution of (1.1) is governed by the self-similar fundamental solution  $U_M(x, t)$ :

$$\lim_{t \rightarrow \infty} t^{n(1-1/p)/\alpha} \|u(t, \cdot) - U_M(t, \cdot)\|_p = 0, \quad \text{for each } p \in [1, \infty], \quad (1.6)$$

with  $M = \|u_0\|_1$  ([4], Theorem 2.2). Analogous results hold for  $\alpha = 2$  ([14], see also [13], [12] for related problems). In the paper, we improve (1.5) and provide some other asymptotics. However, our main result is as follows.

**Theorem 1.1.** *Let one of the conditions (1.3), (1.4) holds. Then, the solution  $u(t, x)$  of (1.1) satisfies*

$$\frac{1}{C} (P_t u_0)(x) \leq u(t, x) \leq C (P_t u_0)(x)$$

for some  $C = C(d, \alpha, u_0) > 1$ .

Since we do not know the exact behaviour of  $u_0$ , we cannot give the precise estimates of  $P_t u_0$ . For example, for  $u_0(x) = \frac{1}{1+|x|^{d+\gamma}}$ , where  $\gamma \in (0, \alpha]$ ,  $P_1 u_0(x) \approx \frac{1}{1+|x|^{d+\gamma}}$  (see,

for example, [9]). In order to prove Theorem 1.1, we introduce a function  $u^*(t, x) = t^{d/\alpha} u(t, t^{1/\alpha} x)$ , which is very convenient to deal with. In particular, the estimates of the  $L^p$  norms of  $u^*(t, \cdot)$  does not depend on  $t$ . It is worth mentioning that the methods used to prove Theorem 1.1 may also be applied in the case when  $u_0 = M\delta_0$  and improve the techniques used in the paper [16].

The paper is organized as follows. In Section 2, we collect some properties of  $p(t, x)$  and introduce Duhamel formula. In Section 3, we show some basic asymptotics of the solution  $u(t, x)$  as  $t \rightarrow 0$  or  $|x| \rightarrow \infty$ . Section 4 is devoted to prove Theorem 1.1. Finally, in Section 5, we give the precise description of asymptotic behaviour of the function  $u(t, x)$ .

## 2. Preliminaries

### 2.1. Notation

For two positive functions  $f, g$  we denote  $f \lesssim g$  whenever there exists a constant  $c > 1$  such that  $f(x) < cg(x)$  for every argument  $x$ . If  $f \lesssim g$  and  $g \lesssim f$  we write  $f \approx g$ . Enumerated constants denoted by capital letters do not change in the whole paper while constant denoted by small letters may change from lemma to lemma. By  $|\cdot|$  we denote the Euclidean norm in  $\mathbb{R}$  and  $\mathbb{R}^d$ .

### 2.2. Properties of $p(t, x)$

The function  $p(t, x)$  was introduced as a fundamental solution of (1.2). We recall that it is a kernel of the stable semigroup  $(P_t f)(x) = \int_{\mathbb{R}^d} p(t, x, w) f(w) dw$ , where  $p(t, x, w) = p(t, x - w)$ . It may be also given by the inverse Fourier transform

$$p(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi, \quad t > 0, \quad x \in \mathbb{R}^d.$$

As a consequence, the following scaling property holds

$$p(t, x) = \lambda^{d/\alpha} p(\lambda t, \lambda^{1/\alpha} x), \quad \lambda > 0. \quad (2.1)$$

Furthermore, estimates of both: the function and its gradient are well-known (see e.g. [5]) and can be expressed by

$$p(t, x, y) \approx \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha}}, \quad (2.2)$$

$$|\nabla_y p(t, x, y)| \approx \frac{t |y - x|}{(t^{1/\alpha} + |y - x|)^{d+2+\alpha}}. \quad (2.3)$$

In particular, we have

$$|b \cdot \nabla_y p(t, x, y)| \lesssim t^{-1/\alpha} p(t, x, y), \quad (2.4)$$

where  $b \in \mathbb{R}^d$  is a constant vector.

### 2.3. Duhamel formula

One of the main tools we use in this paper is the following Duhamel formula

$$u(t, x) = (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) b \cdot \nabla_z [u(s, z)]^{q+1} dz ds. \quad (2.5)$$

Here, we used the fact that  $u(t, x)$  is non-negative. Integrating by parts, we get

$$u(t, x) = (P_t u_0)(x) - \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_z p(t-s, x, z) [u(s, z)]^{q+1} dz ds. \quad (2.6)$$

Let us denote

$$u^*(t, x) = t^{d/\alpha} u(t, t^{1/\alpha} x).$$

We note that  $u^*(t, x) = u^t(1, x)$ , where  $u^\lambda(t, x) = \lambda^{d/\alpha} u(\lambda t, \lambda^{1/\alpha} x)$  is the rescaled solution, cf. (2.1). Although the function  $u^*(t, x)$  depends on time, it plays a similar role as  $U_M(1, x)$  in [16].

Let us observe that under (1.3) or (1.4), we have

$$u(t, x)^{q+1} = u(t, x)^{q_0+1} u(t, x)^{q-q_0} \leq c u(t, x)^{q_0+1},$$

where  $c = 1$  in the case  $q = q_0$  and  $c = \sup_{t>0} \|u_0(\cdot)\|_\infty^{q-q_0}$  in the case  $q > q_0$  (see formula 3.7 in [3]). Now, by scaling property of  $p(t, x)$  and some substitutions in the integrals, we

have

$$\begin{aligned}
t^{d/\alpha} \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_z p(t-s, t^{1/\alpha}x, z) [u(s, z)]^{q_0+1} dz ds & \quad (2.7) \\
&= t^{-1/\alpha} \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_z p(1-s/t, x, t^{-1/\alpha}z) [u(s, z)]^{q_0+1} dz ds \\
&= t^{1-1/\alpha} \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_z p(1-v, x, t^{-1/\alpha}z) [u(vt, z)]^{q_0+1} dz dv \\
&= t^{(d+\alpha-1)/\alpha} \int_0^1 \int_{\mathbb{R}^d} v^{d/\alpha} b \cdot \nabla_w p(1-v, x, v^{1/\alpha}w) [u(vt, (vt)^{1/\alpha}w)]^{q_0+1} dw dv \\
&= \alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1-r^\alpha, x, rw) (rt^{1/\alpha})^{d(q_0+1)} [u(r^\alpha t, rt^{1/\alpha}w)]^{q_0+1} dw dr.
\end{aligned}$$

Finally, we get in both cases (1.3) and (1.4)

$$u^*(t, x) \leq (P_t^* u_0)(x) + c t^{d/\alpha} \int_0^t \int_{\mathbb{R}^d} |b \cdot \nabla_w p(t-s, t^{1/\alpha}x, z)| [u(s, z)]^{q_0+1} dz ds, \quad (2.8)$$

$$\begin{aligned}
&= (P_t^* u_0)(x) + c \alpha \int_0^1 \int_{\mathbb{R}^d} |b \cdot \nabla_w p(1-r^\alpha, x, rw)| [u^*(r^\alpha t, w)]^{q_0+1} dw dr, \\
&\leq (P_t^* u_0)(x) + C_1 \int_0^1 \int_{\mathbb{R}^d} \frac{p(1-r^\alpha, x, rw)}{(1-r^\alpha)^{1/\alpha} + |x-rw|} [u^*(r^\alpha t, w)]^{q_0+1} dw dr, \quad (2.9)
\end{aligned}$$

where  $C_1 = C_1(d, \alpha, u_0)$  and

$$(P_t^* u_0)(x) = t^{d/\alpha} (P_t u_0)(t^{1/\alpha}x).$$

We note that  $P_t^*$  is not a semigroup, we use this notation by the similarity to the definition of  $u^*$ .

### 3. Properties of $u^*$

The function  $u^*(t, x)$  possesses some convenient properties which make it very useful to deal with. First of them is a uniform upper bound of every  $L^p$ -norm.

**Lemma 3.1.** *There exists  $C = C(d) > 0$  such that*

$$\|u^*(t, \cdot)\|_p < C \|u_0\|_1, \quad t > 0, \quad p \in [1, \infty]. \quad (3.1)$$

*Proof.* We base on the formula 3.14 in [3], which implies that for every  $p \in [1, \infty]$  there exists  $C_{d,p} > 1$  such that

$$\|u(t, x)\|_p < C_{d,p} \|u_0\|_1 t^{-d(1-1/p)/\alpha}. \quad (3.2)$$

This directly gives us (3.1) for  $p = \infty$ . For  $p = 1$ , it is enough to substitute  $x = t^{1/\alpha}w$  when computing the  $L^1$  norm. For  $p \in (1, \infty)$ , we use the elementary interpolation inequality and get

$$\|u^*(t, \cdot)\|_p \leq \|u^*(t, \cdot)\|_\infty^{1-1/p} \|u^*(t, \cdot)\|_1^{1/p} \leq C_{d,1} C_{d,\infty} \|u_0\|_1,$$

which ends the proof.  $\square$

The next two propositions show that the function  $u^*(t, x)$  decays uniformly as  $t$  tends to zero or infinity.

**Proposition 3.2.** *Assume (1.3) or (1.4) holds. Then, we have*

$$\lim_{t \rightarrow 0} \|u^*(t, \cdot)\|_\infty = 0.$$

*Proof.* Let  $0 < \varepsilon < 1/2$ . There exists  $R > 0$  such that  $\int_{|u_0| > R} u_0(w) dw < \varepsilon$ . Then, using estimates (2.2) of  $p(t, x)$ , we get

$$\begin{aligned} (P_t^* u_0)(x) &= \int_{|u_0| > R} p(1, x, wt^{-1/\alpha}) u_0(w) dw + t^{d/\alpha} \int_{|u_0| \leq R} p(t, xt^{1/\alpha}, w) u_0(w) dw \\ &\lesssim \varepsilon + R t^{d/\alpha}, \end{aligned}$$

which is small enough for  $t$  close to zero. Now, by (2.4), the integral in (2.8) may be estimated by

$$\begin{aligned} &t^{d/\alpha} \int_0^t \int_{\mathbb{R}^d} |b \cdot \nabla_z p(t-s, xt^{1/\alpha}, z)| [u(s, z)]^{q_0+1} dz ds \\ &\lesssim \int_0^t \int_{\mathbb{R}^d} t^{d/\alpha} (t-s)^{-1/\alpha} p(t-s, xt^{1/\alpha}, z) [u(s, z)]^{q_0+1} dz ds \\ &= \int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t = I_1(t, x) + I_2(t, x). \end{aligned}$$

We start with estimating  $I_2(t, x)$ . By (3.2) with  $p = \infty$ , we have

$$\begin{aligned} I_2(t, x) &\lesssim t^{d/\alpha} \int_{(1-\varepsilon)t}^t (t-s)^{-1/\alpha} \int_{\mathbb{R}^d} p(t-s, xt^{1/\alpha}, z) ((1-\varepsilon)t)^{-d(q_0+1)/\alpha} dz ds \\ &\lesssim t^{-(\alpha-1)/\alpha} \int_0^{\varepsilon t} s^{-1/\alpha} ds = \frac{\varepsilon^{1-1/\alpha}}{1-1/\alpha}. \end{aligned}$$

Furthermore, the formulae (2.4) and (3.2) with  $p = \infty$  imply

$$I_1(t, x) \lesssim \varepsilon^{-1/\alpha} t^{(d-1)/\alpha} \int_0^{(1-\varepsilon)t} s^{-(\alpha-1)/\alpha} \int_{\mathbb{R}^d} p(t-s, xt^{1/\alpha}, z) u(s, z) dz ds.$$

We take  $\tilde{R} > 0$  such that  $\int_{|u_0(x)| > \tilde{R}} |u_0(x)| dx < \varepsilon^{(d+2)/\alpha}$ . Let  $v(t, x)$  be a solution of the problem (1.1) with initial condition  $v(0, x) = \mathbf{1}_{\{|u_0(x)| < \tilde{R}\}} u_0(x)$ . Thus, for every  $t > 0$ , we obtain ([4, Lemma 3.1])

$$\begin{aligned} \|v(t, \cdot) - u(t, \cdot)\|_1 &\leq \|v(0, \cdot) - u(0, \cdot)\|_1 < \varepsilon^{(d+2)/\alpha}, \\ \|v(t, \cdot)\|_\infty &\leq \|v(0, \cdot)\|_\infty \leq \tilde{R}. \end{aligned}$$

Consequently,

$$\begin{aligned} I_1(t, x) &\lesssim \varepsilon^{-1/\alpha} t^{(d-1)/\alpha} \int_0^{(1-\varepsilon)t} s^{-(\alpha-1)/\alpha} \int_{\mathbb{R}^d} p(t-s, xt^{1/\alpha}, z) v(s, z) dz ds \\ &\quad + \varepsilon^{-1/\alpha} t^{(d-1)/\alpha} \int_0^{(1-\varepsilon)t} s^{-(\alpha-1)/\alpha} \int_{\mathbb{R}^d} p(t-s, t^{1/\alpha}x, z) |u(s, z) - v(s, z)| dz ds. \\ &\lesssim \tilde{R} \varepsilon^{-1/\alpha} t^{d/\alpha} + \varepsilon^{-1/\alpha} t^{(d-1)/\alpha} \int_0^{(1-\varepsilon)t} s^{-(\alpha-1)/\alpha} \int_{\mathbb{R}^d} (\varepsilon t)^{-d/\alpha} |u(s, z) - v(s, z)| dz ds. \\ &\lesssim \tilde{R} \varepsilon^{-1/\alpha} t^{d/\alpha} + \varepsilon^{1/\alpha}. \end{aligned}$$

Therefore,  $\lim_{t \rightarrow 0} \|u^*(t, \cdot)\|_\infty \leq c\varepsilon^{1/\alpha}$  for all  $\varepsilon \in (0, 1/2)$ , which ends the proof.  $\square$

**Proposition 3.3.** *Assume (1.3) or (1.4) holds. Then, we have*

$$\lim_{|x| \rightarrow \infty} \|u^*(\cdot, x)\|_\infty = 0.$$

*Proof.* Let  $0 < \varepsilon < 1/2$ . By Proposition 3.2, there exists  $t_0 > 0$  such that  $\|u^*(t, \cdot)\|_\infty < \varepsilon$  for  $t \leq t_0$ . Therefore, we have to consider only  $t > t_0$ . We will show that both terms in (2.9) tends uniformly to zero as  $t \rightarrow 0$ . Since  $u_0 \in L^1$ , there is a radius  $R > 0$  such that  $\int_{|x| > R} u_0(x) dx < \varepsilon$ . Then, by (2.2), we get for  $|x| > R/t_0^\alpha$

$$\begin{aligned} (P_t^* u_0)(x) &= \int_{|w| > R} + \int_{|w| \leq R} p(1, x, t^{-1/\alpha} w) u_0(w) dw \\ &\lesssim \varepsilon + \frac{\|u_0\|_1}{(|x| - R/t_0^{1/\alpha})^{d+\alpha}}, \end{aligned}$$



which is small for large  $|x|$ . In order to estimate the integral in (2.9) we divide it as follows

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}^d} \frac{p(1-r^\alpha, x, rw)}{(1-r^\alpha)^{1/\alpha} + |x-rw|} [u^*(r^\alpha t, w)]^{q_0+1} dw dr \\ &= \int_0^\varepsilon + \int_\varepsilon^{(1-\varepsilon)^{1/\alpha}} + \int_{(1-\varepsilon)^{1/\alpha}}^1 := I_1 + I_2 + I_3. \end{aligned}$$

Applying (2.2) and (3.1) for  $p = 1 + q_0$ , we obtain

$$\begin{aligned} I_1 &\lesssim (1-\varepsilon^\alpha)^{-(d+1)/\alpha} \int_0^\varepsilon \int_{\mathbb{R}^d} [u^*(r^\alpha t, w)]^{q_0+1} dw dr \\ &\lesssim \varepsilon \|u_0\|_1^{q_0+1}. \end{aligned}$$

Next, by (3.1) for  $p = \infty$ ,

$$\begin{aligned} I_3 &\lesssim \|u_0\|_1^{q_0+1} \int_{(1-\varepsilon)^{1/\alpha}}^1 (1-r^\alpha)^{-1/\alpha} \int_{\mathbb{R}^d} p(1-r^\alpha, x, rw) dw dr \\ &= \|u_0\|_1^{q_0+1} \int_{(1-\varepsilon)^{1/\alpha}}^1 (1-r^\alpha)^{-1/\alpha} r^{-d} dr \\ &= \frac{1}{\alpha} \|u_0\|_1^{q_0+1} \int_0^\varepsilon s^{-1/\alpha} (1-s)^{1/\alpha-d-1} ds \\ &\lesssim \|u_0\|_1^{q_0+1} \varepsilon^{1-1/\alpha}. \end{aligned}$$

Now we are going to deal with the integral  $I_2$ . By [4, Lemma 3.10], we have

$$\lim_{R \rightarrow \infty} \sup_{t > T} \int_{|x| > R} u^*(t, x) dx = 0, \quad \text{for every } T > 0.$$

Hence, there exists  $R > 0$  such that  $\int_{|w| > R} u^*(s, w) dw < \varepsilon^{(d+1+\alpha)/\alpha}$  for every  $s > \varepsilon^{1/\alpha} t_0$ .

Thus, for  $|x| > R$ , we get

$$\begin{aligned} I_2 &\lesssim \|u_0\|_1^{q_0} \int_{(\varepsilon, (1-\varepsilon)^{1/\alpha})} \int_{|w| > R} (1-r^\alpha)^{-(d+1)/\alpha} u^*(r^\alpha t, w) dw dr \\ &\quad + \int_{(\varepsilon, (1-\varepsilon)^{1/\alpha})} \int_{|w| \leq R} \frac{1}{(x-rw)^{d+1+\alpha}} [u^*(r^\alpha t, w)]^{1+q_0} dw dr \\ &\lesssim \|u_0\|_1^{q_0} \varepsilon^{-(d+1)/\alpha} \int_0^1 \int_{|w| > R} u^*(r^\alpha t, w) dw dr + \int_0^1 \int_{\mathbb{R}^d} \frac{[u^*(r^\alpha t, w)]^{1+q_0}}{(|x|-R)^{d+1+\alpha}} dw dr \\ &\lesssim \varepsilon \|u_0\|_1^{q_0} + \frac{\|u_0\|_1^{1+q_0}}{(|x|-R)^{d+1+\alpha}}, \end{aligned}$$

which is small enough for sufficiently large  $|x|$ . This ends the proof.  $\square$

*Remark 3.4.* Using the same methodology, a noticeably simpler proof of Proposition 3.2 in [16] may be obtained.

#### 4. Main results

The main goal of this section is to prove Theorem 1.1. Additionally, we present some asymptotics of the function  $u^*(t, x)$ , which play a crucial role in proof of the main theorem. Nevertheless, they are also interesting as separate results, which is discussed in Section 5, where asymptotics of  $u(t, x)$  are studied.

To shorten notation, we denote for  $\beta \in [0, 1)$

$$h_\beta(r, x, w) = r^{-\beta}(1 - r^\alpha)^{-1/\alpha}p(1 - r^\alpha, x, rw). \quad (4.1)$$

The below-given technical lemma is intensively exploit in proofs of Theorems 4.2 and 4.3.

**Lemma 4.1.** *Let  $\beta \in (0, 1)$  and  $f : \mathbb{R}^d \rightarrow [0, \infty)$ ,  $g : [0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$  be such that integrals in (4.2) and (4.3) converge. There exist  $C_2 = C_2(d, \alpha, \beta)$  and  $C_3 = C_3(d, \alpha, \beta)$  such that*

$$(i) \quad \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, w) (P_{r^\alpha t}^* f)(w) dw dr = C_2 (P_t^* f)(x), \quad (4.2)$$

$$(ii) \quad \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, w) \int_0^1 h_0(s, w, z) g(s^\alpha r^\alpha t, z) ds dw dr \\ < C_3 \int_0^1 h_\beta(r, x, z) g(r^\alpha t, z) dr, \quad (4.3)$$

where  $t > 0$ ,  $z \in \mathbb{R}^d$ .

*Proof.* We note that for any  $s, t, \beta, \gamma \in (0, \infty)$  and  $x, z \in \mathbb{R}^d$ , by scaling property (2.1) of  $p(t, x, y)$ , we get

$$\begin{aligned} \int_{\mathbb{R}^d} p(s, x, \beta w) p(t, \gamma w, z) dw &= \int_{\mathbb{R}^d} \beta^{-d} p(\beta^{-\alpha} s, \beta^{-1} x, w) \gamma^{-d} p(\gamma^{-\alpha} t, w, \gamma^{-1} z) dw \\ &= (\beta \gamma)^{-d} p(\beta^{-\alpha} s + \gamma^{-\alpha} t, \beta^{-1} x, \gamma^{-1} z) dw \\ &= p(\gamma^\alpha s + \beta^\alpha t, \gamma x, \beta z). \end{aligned} \quad (4.4)$$

Since

$$(P_s^* f)(w) = \int_{\mathbb{R}^d} s^{d/\alpha} p(s, s^{1/\alpha} w, z) f(z) dz = \int_{\mathbb{R}^d} p(1, w, s^{-1/\alpha} z) f(z) dz,$$

(4.4) gives us

$$\begin{aligned}
& \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) (P_{r^\alpha t}^* u_0)(w) dw \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) p(1, w, (rt^{1/\alpha})^{-1} z) u_0(z) dw dz \\
&= \int_{\mathbb{R}^d} p(1, x, t^{-1/\alpha} z) u_0(z) dz = (P_t^* u_0)(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, w) (P_{r^\alpha t}^* f)(w) dw dr \\
&= (P_t^* u_0)(x) \int_0^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} dr = C_2 (P_t^* f)(x),
\end{aligned}$$

which proves (i). Furthermore, by (4.4), we have

$$\int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) p(1 - s^\alpha, w, sz) dw = p(1 - (rs)^\alpha, x, rsz).$$

Hence, substituting  $s = v/r$  in the second line, we get

$$\begin{aligned}
& \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, w) \int_0^1 h_0(s, w, z) g(s^\alpha r^\alpha t, z) ds dw dr \\
&= \int_0^1 \int_0^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} (1 - s^\alpha)^{-1/\alpha} p(1 - (rs)^\alpha, x, rsz) g(s^\alpha r^\alpha t, z) ds dr \\
&= \int_0^1 \int_0^r r^{-\beta} (1 - r^\alpha)^{-1/\alpha} (r^\alpha - v^\alpha)^{-1/\alpha} p(1 - v^\alpha, x, vz) g(v^\alpha t, z) dv dr \\
&= \int_0^1 p(1 - v^\alpha, x, vz) g(v^\alpha t, z) \int_v^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} (r^\alpha - v^\alpha)^{-1/\alpha} dr dv.
\end{aligned}$$

Using the estimate ([16], Corollary 4.3)

$$\int_v^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} (r^\alpha - v^\alpha)^{-1/\alpha} dr \lesssim v^{-\beta} (1 - v)^{-1/\alpha},$$

we obtain the assertion (ii). □

Theorems 4.2 and 4.3 show that the distance between  $u^*(t, x)$  and  $P_t^* u_0$  tends to zero as  $t \rightarrow 0$  or  $|x| \rightarrow \infty$ . To avoid repeating long integrals in the proofs of those theorems, we rewrite (2.9) as

$$u^*(t, x) \leq P_t^* u_0(x) + I(t, x), \tag{4.5}$$

where

$$0 \leq I(t, x) \leq C_1 \int_0^1 \int_{\mathbb{R}^d} \frac{p(1 - r^\alpha, x, rw)}{(1 - r^\alpha)^{1/\alpha} + |x - rw|} [u^*(r^\alpha t, w)]^{q_0+1} dw dr. \quad (4.6)$$

**Theorem 4.2.** *Assume (1.3) or (1.4) holds. We have*

$$\lim_{t \rightarrow 0} \left\| \frac{u^*(t, \cdot)}{(P_t^* f)(\cdot)} - 1 \right\|_\infty = 0. \quad (4.7)$$

*Proof.* First, we estimate the integral  $I(t, x)$  from (4.6) as follows

$$0 \leq I(t, x) \leq C_1 \int_0^1 \int_{\mathbb{R}^d} (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw) [u^*(r^\alpha t, w)]^{q_0+1} dw dr.$$

Let  $0 < \eta, \beta < 1$ . By Proposition 3.2, we may choose  $t_0$  such that

$$u^*(t, x) < \left( \frac{\eta}{C_1(C_2 \vee C_3)} \right)^{1/q_0}, \quad \text{for } t < t_0 \text{ and } x \in \mathbb{R}^d, \quad (4.8)$$

where  $C_2$  and  $C_3$  are the constants from Lemma 4.1. We will show that

$$I(t, x) \leq \frac{\eta}{1 - \eta} (P_t^* u_0)(x), \quad t < t_0, \quad x \in \mathbb{R}^d. \quad (4.9)$$

Let  $t < t_0$ . Then, using notation introduced in (4.1),

$$I(t, x) \leq \frac{\eta}{C_2 \vee C_3} \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, w) u^*(r^\alpha t, w) dw dr := J(t, x). \quad (4.10)$$

We note that by (4.5), we have

$$u^*(r^\alpha t, w) \leq (P_{r^\alpha t}^* u_0)(w) + C_1 \int_0^1 \int_{\mathbb{R}^d} h_0(s, w, z) [u^*(s^\alpha r^\alpha t, z)]^{q_0+1} dz ds. \quad (4.11)$$

We apply (4.11) to (4.10) and, by Lemma 4.1 and (4.8), we get

$$\begin{aligned} J(t, x) &\leq \eta (P_t^* u_0)(x) + C_1 \eta \int_0^1 \int_{\mathbb{R}^d} h_\beta(r, x, z) [u^*(r^\alpha t, z)]^{q_0+1} dz dr, \\ &\leq \eta (P_t^* u_0)(x) + \eta J(t, x). \end{aligned} \quad (4.12)$$

Hence,  $(1 - \eta)J(t, x) \leq \eta (P_t^* u_0)(x)$  and, by (4.10), we get (4.9).

Consequently, for  $t < t_0$  and  $x \in \mathbb{R}^d$ ,  $|u^*(t, x) - P_t^* u_0(x)| \leq \frac{\eta}{1 - \eta} P_t^* u_0(x)$ , which is equivalent to

$$\left\| \frac{u^*(t, \cdot)}{(P_t^* u_0)(\cdot)} - 1 \right\|_\infty \leq \frac{\eta}{1 - \eta}, \quad 0 < t < t_0.$$

The proof is completed.  $\square$

Using a similar method we get the asymptotics of  $u^*(t, x)$  for  $|x| \rightarrow \infty$ .

**Theorem 4.3.** *Assume (1.3) or (1.4) hold. We have*

$$\lim_{|x| \rightarrow \infty} \sup_{t > 0} \left| 1 - \frac{u^*(t, x)}{(P_t^* f)(x)} \right| = 0. \quad (4.13)$$

*Proof.* Let  $0 < \eta, \beta < 1$ . By Proposition 3.3 we may choose  $R > 0$  such that  $u^*(t, x) < \left( \frac{\eta}{C_1(C_2 \vee C_3)} \right)^{1/q_0}$  for  $|x| > R$  and  $t > 0$ . We divide the integral  $I(t, x)$  from (4.6) into  $\int_0^1 \int_{|w| \leq R} + \int_0^1 \int_{|w| > R}$  and estimate it as follows

$$\begin{aligned} I(t, x) &\leq C_1 \int_0^1 \int_{|w| < R} \frac{p(1 - r^\alpha, x, rw)}{|x - rw|} [u^*(r^\alpha t, w)]^{1+q_0} dw dr \\ &\quad + \frac{\eta}{C_2 \vee C_3} \int_0^1 \int_{|w| > R} h_\beta(r, x, w) u^*(r^\alpha t, w) dw dr. \end{aligned} \quad (4.14)$$

Similarly, we get

$$\begin{aligned} u^*(r^\alpha t, w) &\leq P_{r^\alpha t}^* u_0(x) + \int_0^1 \int_{|z| \leq R} h_0(s, w, z) [u^*(s^\alpha r^\alpha t, z)]^{1+q_0} dz ds \\ &\quad + \frac{\eta}{C_2 \vee C_3} \int_0^1 \int_{|z| > R} h_0(s, w, z) u^*(s^\alpha r^\alpha t, z) dz ds. \end{aligned} \quad (4.15)$$

First, we will estimate the last expression in (4.14)

$$J(t, x) = \frac{\eta}{C_2 \vee C_3} \int_0^1 \int_{|w| > R} h_\beta(r, x, w) u^*(r^\alpha t, w) dw dr. \quad (4.16)$$

We put (4.15) into (4.16) and, by virtue of Lemma 4.1, we get

$$J(t, x) \leq \eta (P_t^* u_0)(x) + \eta \int_0^1 \int_{|w| < R} h_\beta(r, x, w) [u^*(r^\alpha t, w)]^{1+q_0} dw dr + \eta J(t, x).$$

Hence,

$$J(t, x) \leq \frac{\eta}{1 - \eta} (P_t^* u_0)(x) + \frac{\eta}{1 - \eta} \int_0^1 \int_{|w| < R} h_\beta(r, x, w) [u^*(r^\alpha t, w)]^{1+q_0} dw dr. \quad (4.17)$$

Then, for  $|x| > (2R) \vee \frac{1}{\eta}$  by (4.14) and (4.17), we get

$$\begin{aligned} I(t, x) &\leq c_1 \int_0^1 \int_{|w| < R} \left( \frac{2}{|x|} \right)^{d+\alpha+1} [u^*(r^\alpha t, w)]^{1+q_0} dw dr + \frac{\eta}{1 - \eta} (P_t^* u_0)(x) \\ &\quad + \frac{\eta}{1 - \eta} c_1 \int_0^1 \int_{|w| < R} r^{-\beta} (1 - r^\alpha)^{-1/\alpha} \left( \frac{2}{|x|} \right)^{d+\alpha} [u^*(r^\alpha t, w)]^{1+q_0} dw dr \\ &\leq \frac{\eta}{1 - \eta} (P_t^* u_0)(x) + \frac{\eta}{1 - \eta} \frac{c_2}{|x|^{d+\alpha}}, \end{aligned}$$

for some  $c_2 = c_2(d, \alpha, \beta, u_0) > 0$ . The next step is to prove that  $(P_t^* u_0)(x) \gtrsim \frac{1}{|x|^{d+\alpha}}$  for large  $|x|$  and  $t$  bounded away from zero. There is  $r_0 > 0$  such that  $\int_{|w| < r_0} u_0(w) dw > \|u_0\|_1/2$ . Let  $t_0 > 0$ . For  $t > t_0$  and  $x \in \mathbb{R}^d$  we get

$$\begin{aligned} (P_t^* u_0)(x) &\geq \int_{B(0, r_0)} p(1, x, t^{-1/\alpha} w) u_0(w) dw \\ &\gtrsim \frac{1}{(1 + |x| + r_0/t_0^{1/\alpha})^{d+\alpha}} \int_{|w| < r_0} u_0(w) dw \gtrsim \frac{\|u_0\|_1}{1 \vee |x|^{d+\alpha}}. \end{aligned} \quad (4.18)$$

Combining all together, there exists  $c_3 = c_3(d, \alpha, \beta, u_0) > 0$  such that

$$|I(t, x)| \leq c_3 \frac{\eta}{1 - \eta} (P_t^* u_0)(x)$$

holds whenever  $|x| > (2R) \vee \frac{1}{\eta}$  and  $t > t_0$ . Consequently

$$\left| 1 - \frac{u^*(t, x)}{(P_t^* u_0)(x)} \right| < c_3 \frac{\eta}{1 - \eta}.$$

Now, applying Theorem 4.2, we get the above inequality for  $t_0 = 0$ , which ends the proof.  $\square$

Finally, we are prepared to prove the main result.

*Proof of Theorem 1.1.* The equivalent statement of the theorem is

$$u^*(t, x) \approx P_t^* u_0(x), \quad t > 0, x \in \mathbb{R}^d$$

Theorems 4.2 and 4.3 imply that there exist  $R > 0$  and  $t_0 > 0$  such that the required estimates hold whenever  $t \in (0, t_0)$  or  $|x| > R$ . What has left is to consider  $(t, |x|) \in [t_0, \infty) \times [0, R]$ . Observe that by (4.18) and (2.2), we have

$$c_1 \|u_0\|_1 \leq P_t^* u_0(x) \leq c_2 \|u_0\|_1, \quad (t, |x|) \in [t_0, \infty) \times [0, R], \quad (4.19)$$

for some constants  $c_1, c_2 > 0$  ( $c_1$  depends on  $t_0$  and  $R$ ). To end the proof, we have to show that  $u^*(t, x) \approx 1$  for  $t \geq t_0$  and  $|x| \leq R$ . The upper bound comes from (3.1). Next, under assumptions (1.3) or (1.4), by (3.2), we have

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^d} |b \cdot \nabla_w p(1 - r^\alpha, x, rw)| [u^*(r^\alpha t, w)]^{q_0+1} dw dr \\ &\leq c_3 \int_0^{1/2} \int_{\mathbb{R}^d} [u^*(r^\alpha t, w)]^{1+q_0} dw dr + \int_{1/2}^1 \int_{\mathbb{R}^d} |\nabla_w p(1 - r^\alpha, x, rw)| \|u_0\|_1^{1+q_0} dw dr \\ &\leq c_4 \|u_0\|_1^{1+q_0} \end{aligned} \quad (4.20)$$

for some constant  $c_4 = c_4(d, \alpha) > 0$ .

Now, let  $\varepsilon \in (0, 1)$  and let  $u_\varepsilon(t, x)$  be the solution of (1.1) with the initial condition  $u_\varepsilon(0, x) = \varepsilon u_0(x)$ . Put  $u_\varepsilon^*(t, x) = t^{d/\alpha} u_\varepsilon(t, t^{1/\alpha} x)$ . Then, we have for every  $t > 0$ ,  $\|u_\varepsilon^*(t, \cdot)\|_\infty \leq \varepsilon \|u_0\|_1$  and  $\|u_\varepsilon^*(t, \cdot)\|_1 \leq \varepsilon \|u_0\|_1$ . Thus, by (4.19) and (4.20),

$$\begin{aligned} u_\varepsilon^*(t, x) &\gtrsim \varepsilon P_t^* u_0(x) - \int_0^1 \int_{\mathbb{R}^d} |b \cdot \nabla_w p(1 - r^\alpha, x, rw)| [u_\varepsilon^*(r^\alpha t, w)]^{1+q_0} dw dr \\ &\geq \varepsilon c_1 \|u_0\|_1 - \varepsilon^{1+q_0} c_4 \|u_0\|_1^{1+q_0}, \end{aligned}$$

for  $t \geq t_0$  and  $|x| \leq R$ . Taking  $\varepsilon = \left(\frac{c_1}{2c_3}\right)^{1/q_0} \|u_0\|_1^{-1}$ , we get  $u_\varepsilon^*(t, x) \geq \varepsilon c_5 \|u_0\|_1 > 0$ . Since solutions of (1.1) preserve the order of initial conditions (see [4], Lemma 3.1), we have  $u^*(t, x) > u_\varepsilon^*(t, x)$ , and the proof is complete.  $\square$

## 5. Asymptotic behaviour of solutions

It is easy to see, that

$$\lim_{t \rightarrow \infty} t^{n(1-1/p)/\alpha} \|(P_t u_0)(\cdot) - Mp(t, \cdot)\|_p = 0 \quad (5.1)$$

holds for every  $p \in [1, \infty]$  and  $u_0 \in L^1$ . Applying this to (1.5), we obtain

$$\lim_{t \rightarrow \infty} t^{n(1-1/p)/\alpha} \|u(t, \cdot) - Mp(t, \cdot)\|_p = 0.$$

This form of the result is presented e.g. in [14], where  $\alpha = 2$  is considered. It seems to be more useful than (1.5), since the function  $p(t, x)$  is well known and does not depend on  $u_0$ . Such formulation is also a more natural counterpart of (1.6). Nevertheless, it may be concluded from Theorem 1.1 that we have to employ the function  $P_t u_0$  to describe the behaviour of  $u(t, x)$  more precisely. In the sequel, we discuss asymptotics of the quotient  $u(t, x)/(P_t u_0)(x)$ . We also give another improvement of (1.5). Some results are already provided in Section 4. In particular, Proposition 4.2 is equivalent to the following equality.

**Corollary 5.1.** *Under (1.3) or (1.4) we have*

$$\lim_{t \rightarrow 0} \left\| \frac{u(t, \cdot)}{(P_t u_0)(\cdot)} - 1 \right\|_\infty = 0$$

Theorem 4.3 could be also reformulated in language of the function  $u(t, x)$ , but it would lose its clear form. Additionally, a stronger and clearer result, under condition (1.4), will be given at the end of this section. Before that, we discuss the large time behaviour of the solution of (1.1) with this condition.

**Proposition 5.2.** *Assume (1.4) holds. For every  $0 < \gamma < (d(q - q_0) \wedge 1)/\alpha$ , we have*

$$\lim_{t \rightarrow \infty} t^\gamma \left\| 1 - \frac{u(t, \cdot)}{(P_t u_0)(\cdot)} \right\|_\infty = 0.$$

*Proof.* There exists  $\varepsilon > 0$  such that  $\gamma + \varepsilon < (d(q - q_0) \wedge 1)/\alpha$ . Additionally, using (1.4), we have

$$(P_s u_0)(z) \lesssim (s^{-d/\alpha} \|u_0\|_1) \wedge \|u_0\|_\infty.$$

Consequently, since  $q > q_0 + \alpha(\gamma + \varepsilon)/d$ , we get

$$[(P_s u_0)(z)]^q \lesssim [(P_s u_0)(z)]^{q_0 + \frac{\alpha(\gamma + \varepsilon)}{d}} \lesssim s^{-\frac{d}{\alpha}(q_0 + \frac{\alpha(\gamma + \varepsilon)}{d})} = s^{-\gamma - \varepsilon - (\alpha - 1)/\alpha}.$$

Then, by Theorem 1.1, we obtain

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_w p(t - s, x, z) [u(s, z)]^{q+1} dz ds \right| \\ & \lesssim \int_0^t \int_{\mathbb{R}^d} (t - s)^{-1/\alpha} s^{-\gamma - \varepsilon - (\alpha - 1)/\alpha} p(t - s, x, z) (P_s u_0)(z) dz ds \\ & = t^{-\gamma - \varepsilon} \int_0^1 \int_{\mathbb{R}^d} (1 - r)^{-1/\alpha} r^{-\gamma - \varepsilon - (\alpha - 1)/\alpha} p(t, x, w) u_0(w) dw ds \\ & = c t^{-\gamma - \varepsilon} P_t u_0(x). \end{aligned} \tag{5.2}$$

The last integral is finite whenever  $-\gamma - \varepsilon - (\alpha - 1)/\alpha > -1$ , which explains the importance of the assumption  $\gamma + \varepsilon < 1/\alpha$ . Finally, by (2.6) we arrive at

$$t^\gamma \left\| 1 - \frac{u(t, \cdot)}{(P_t u_0)(\cdot)} \right\|_\infty \lesssim t^{-\varepsilon}, \quad x \in \mathbb{R}^d, \quad t > 0.$$

The proof is complete. □

*Remark 5.3.* A result of that kind cannot be obtained in the case  $q = q_0$ , since (1.6) and (5.1) hold and

$$\left\| 1 - \frac{U_M(t, \cdot)}{Mp(t, \cdot)} \right\|_\infty = \left\| 1 - \frac{U_M(1, \cdot)}{Mp(1, \cdot)} \right\|_\infty \neq 0,$$

which follows from scaling properties of the functions  $p(t, x)$  and  $U_M(t, x)$  (see [4], Theorem 2.1).



The following result gives better large time asymptotics of  $u(t, x)$  than one can obtain from [3] (see (1.5)).

**Corollary 5.4.** *Assume (1.4) holds. For every  $0 < \gamma < (d(q - q_0) \wedge 1)/\alpha$ , we have*

$$\lim_{t \rightarrow \infty} t^{\gamma + d(1-1/p)/\alpha} \|u(t, \cdot) - (P_t u_0)(\cdot)\|_p = 0.$$

*Proof.* Let, as in the proof of Proposition 5.2,  $\varepsilon > 0$  such that  $\gamma + \varepsilon < (d(q - q_0) \wedge 1)/\alpha$ . Then, (5.2) gives us

$$|u(t, x) - (P_t u_0)(x)| \lesssim ct^{-\gamma - \varepsilon} (P_t u_0)(x).$$

By Young inequality,  $\|(P_t u_0)(\cdot)\|_p \leq \|p(t, \cdot)\|_p \|u_0\|_1 = ct^{-d(1-1/p)/\alpha} \|u\|_1$ . Hence,

$$\lim_{t \rightarrow \infty} t^{\gamma + d(1-1/p)/\alpha} \|u(t, \cdot) - (P_t u_0)(\cdot)\|_p = 0.$$

□

Combining Theorem 4.3 and Proposition 5.2, we obtain the uniform asymptotics for large  $|x|$ .

**Corollary 5.5.** *Under assumption (1.4), we have*

$$\lim_{|x| \rightarrow \infty} \sup_{t > 0} \left| \frac{u(t, x)}{(P_t u_0)(x)} - 1 \right| = 0.$$

*Proof.* Fix  $\varepsilon > 0$ . In view of Proposition 5.2, there exists  $t_0 > 0$  such that

$$\left| \frac{u(t, x)}{(P_t u_0)(x)} - 1 \right| < \varepsilon, \tag{5.3}$$

whenever  $t > t_0$ . Furthermore, by Theorem 4.3, there is  $R$  such that (5.3) holds if  $|x| > R t^{1/\alpha}$ . Consequently, (5.3) is true for  $t > 0$  and  $|x| > R t_0^{1/\alpha}$ , which ends the proof. □

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